4.2 Null Spaces, Column Spaces, Row Spaces, and Linear Transformations

The Null Space of a Matrix

Definition (null space)

The **null space** of an $m \times n$ matrix A, written as $\operatorname{Nul} A$, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\operatorname{Nul} A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

A more dynamic description of $\operatorname{Nul} A$ is the set of all $\mathbf{x}$ in $\mathbb{R}^n$ that are mapped into the zero vector of $\mathbb{R}^m$ via the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$. See the following figure.

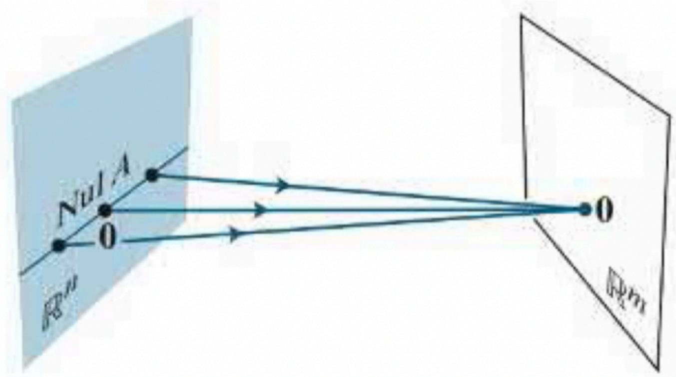

FIGURE 1

Example 1. Determine if $\mathbf{w} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ is in Nul A, where $A = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix}$.

ANS : $A\vec{w} = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

thus $\vec{w} \in \operatorname{Nul} A$

Theorem 2.

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Example 3. In the following exercises, either use an appropriate theorem to show that the given set, W, is a vector space, or find a specific example to the contrary.

(1) $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a+b+c=2 \right\}$

Note W is a subset of $\mathbb{R}^3$. If W is a vector space, the W is a subspace of $\mathbb{R}^3$. But since the zero vector is not W, W is not a vector space.

(2) $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{matrix} a+3b=c \\ b+c+a=d \end{matrix} \right\} \Leftrightarrow \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{matrix} a+3b-c=0 \\ b+c+a-d=0 \end{matrix} \right\}$

The set W is the set of all the solutions to the homogeneous eqn $\begin{cases} a+3b-c=0 \\ a+b+c-d=0 \end{cases}$. Thus $W = \text{Nul } A$, where $A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

Thus W is a subspace of $\mathbb{R}^4$ by Thm 2, and it is a vector space.

(3) $\left\{ \begin{bmatrix} b-5d \\ 2b \\ 2d+1 \\ d \end{bmatrix} : b, d \text{ real} \right\}$

The set W is a subset of $\mathbb{R}^4$. If W is a vector space, then it would be a subspace of $\mathbb{R}^4$. But W is not a subspace of $\mathbb{R}^4$ since the zero vector is not in W. Thus W is not a vector space.

An Explicit Description of Nul A

Example 3. Find an explicit description of $\mathrm{Nul}A$ by listing vectors that span the null space.

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

ANS: First we find the general solution to $A\vec{x}=\vec{0}$.

$$[A \ \ \vec{0}] = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Basic varibles: x_1, x_3, x_5. Free varibles: x_2, x_4

$$\begin{cases} x_1 = 2x_2 - 4x_4 \\ x_3 = 9x_4 \\ x_5 = 0 \end{cases}$$

So

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 - 4x_4 \\ x_2 \\ 9x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix}$$

and a spanning set for Nul$A$ is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix} \right\}$

The Column Space of a Matrix

Definition The **column space** of an $m \times n$ matrix A, written as $\mathrm{Col}\, A$, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then

$$\mathrm{Col}\, A = \mathrm{Span}\,\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$$

Theorem 3.

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

In set notation,

$$\boxed{\mathrm{Col}\, A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}}$$

The notation $A\mathbf{x}$ for vectors in $\mathrm{Col}\, A$ also shows that $\mathrm{Col}\, A$ is the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$
.

Recall from Theorem 4 in Section 1.4 that the columns of A span $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$. We can restate this fact as follows:

Theorem The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

Example 4. Find A such that the given set is $\mathrm{Col}\, A$.

$$\left\{ \begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix} : r, s, t \text{ real} \right\}$$

ANS: An element in the given set can be written

as $r\begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} r \\ s \\ t \end{bmatrix}$

where $r, s, t \in \mathbb{R}$.

So the set is Col A when $A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix}$

Example 5. For the matrix given, (a) find k such that $\mathrm{Nul}A$ is a subspace of $\mathbb{R}^k$, and (b) find k such that $\mathrm{Col}\,A$ is a subspace of $\mathbb{R}^k$.

$$A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 0 & -5 \\ 0 & -5 & 7 \\ -5 & 7 & -2 \end{bmatrix}$$

The matrix A is a 4x3 matrix. So

(a) Nul A is a subspace of $\mathbb{R}^3$.

(b) Col A is a subspace of $\mathbb{R}^4$

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	**Col A**
1. Nul A is a subspace of $\mathbb{R}^n$.	1. $\mathrm{Col}\,A$ is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition $(A\mathbf{x} = \mathbf{0})$ that vectors in Nul A must satisfy.	2. $\mathrm{Col}\,A$ is explicitly defined; that is, you are told how to build vectors in $\mathrm{Col}\,A$.
3. It takes time to find vectors in Nul A. Row operations on $[A \quad \mathbf{0}]$ are required.	3. It is easy to find vectors in $\mathrm{Col}\,A$. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	4. There is an obvious relation between $\mathrm{Col}\,A$ and the entries in A, since each column of A is in $\mathrm{Col}\,A$.
5. A typical vector $\mathbf{v}$ in $\mathrm{Nul}\,A$ has the property that $A\mathbf{v} = \mathbf{0}$.	5. A typical vector $\mathbf{v}$ in $\mathrm{Col}\,A$ has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	6. Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in $\mathrm{Col}\,A$. Row operations on $[A \quad \mathbf{v}]$ are required.
7. Nul $A = \{0\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	7. $\mathrm{Col}\,A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. $\mathrm{Nul}\,A = \{0\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	8. $\mathrm{Col}\,A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

Exercise 6. Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Determine if $\mathbf{w}$ is in Col A. Is w in Nul A ?

Solution. Consider the system with augmented matrix $\begin{bmatrix} A & \mathbf{w} \end{bmatrix}$. Since $\begin{bmatrix} A & \mathbf{w} \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$, the system is consistent and $\mathbf{w}$ is in $\operatorname{Col} A$. Also, since $A\mathbf{w} = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{w}$ is in Nul A.

Exercise 7. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Given a subspace Z of W, let U be the set of all $\mathbf{x}$ in V such that $T(\mathbf{x})$ is in Z. Show that U is a subspace of V.

Solution.

- Since Z is a subspace of $W, \mathbf{0}_W$ is in Z. Since T is linear, $T\left(\mathbf{0}_V\right) = \mathbf{0}_W$. So $\mathbf{0}_V$ is in U.
- Let $\mathbf{x}$ and $\mathbf{y}$ be typical elements in U. Then $T(\mathbf{x})$ and $T(\mathbf{y})$ are in Z, and since Z is a subspace of $W, T(\mathbf{x}) + T(\mathbf{y})$ is also in Z. Since T is linear, $T(\mathbf{x}) + T(\mathbf{y}) = T(\mathbf{x} + \mathbf{y})$. So $T(\mathbf{x} + \mathbf{y})$ is in Z, and $\mathbf{x} + \mathbf{y}$ is in U. Thus U is closed under vector addition.
- Let c be any scalar. Then since $\mathbf{x}$ is in $U, T(\mathbf{x})$ is in Z. Since Z is a subspace of $W, cT(\mathbf{x})$ is also in Z. Since T is linear, $cT(\mathbf{x}) = T(c\mathbf{x})$ and $T(cx)$ is in $T(U)$. Hence U Thus cx is in U and U is closed under scalar multiplication.is a subspace of V.